# Unit 5

OMR Sheet No.__________ Registration No. ①

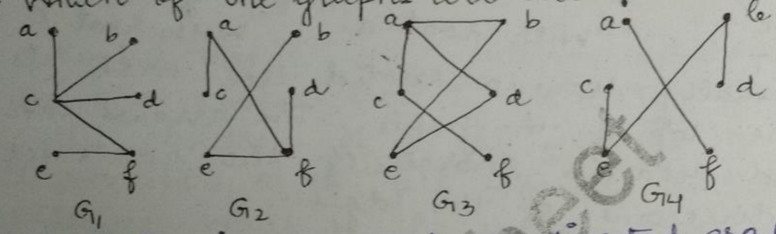
Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

## Trees

**Def:** A tree is a connected undirected graph with no simple circuits.

* Because a tree can't have a simple circuit, a tree can't contain multiple edges or loops.
∴ Any tree must be a simple graph.

Eg Which of the graphs are trees?

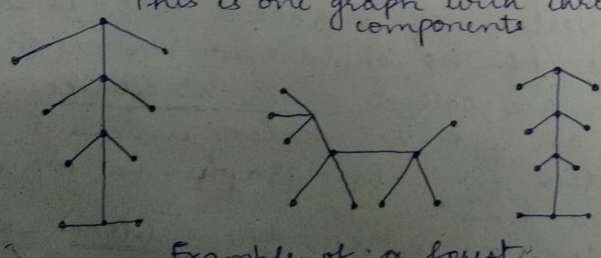



$G_1$ and $G_2$ is connected undirected graph with no simple circuits.
In $G_3$, a, b, e, d, a is a simple circuit so it is not a tree.
$G_4$ is not connected so it is not a tree.

* Any connected graph with no simple circuit is called tree. Graphs containing no simple circuits that are not necessarily connected are called forests. They have the property that each of their connected components is a tree.

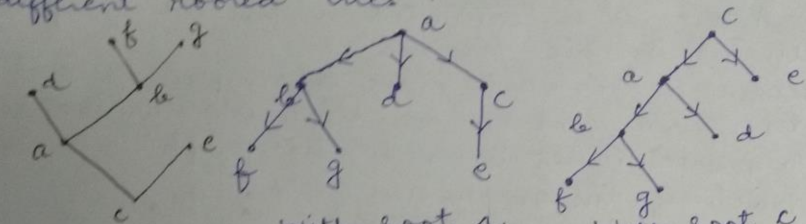
**Theorem:-** An undirected graph is a tree iff there is a unique simple path between any two of its vertices.

This is one graph with three connected components



Example of a forest

Def: A rooted tree is a tree in which ② one v
has been ~~assigned~~ designated as the root a
every edge is directed away from the root.
* Different choices of the root produce different rooted trees.




T　　　With root a　　　With root c.

We usually draw a rooted tree with its root at the top of the graph. The arrows can be omitted because choice of the roote determines the directions of the edges.

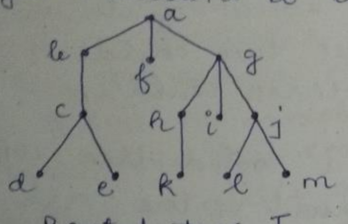
* Suppose that T is a rooted tree. If v is a vertex in T other than the root, the parent of v is the unique vertex u such that there is a directed edge from u to v. When u is parent of v, v is called a child of u. Vertices with same parent are called siblings. The ancestors of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root (i.e. its parent, its parent's parent and so on until the root is reached). The descendants of a vertex v are those vertices that have v as an ancestor. A vertex of a rooted tree is called a leaf if it has no children. Vertices that have children are called internal vertices. The root is an internal vertex unless it is the only vertex in the graph, in which case it is a leaf.

If a is a vertex in a tree, the subtree with a as its root is the subgraph of the tree

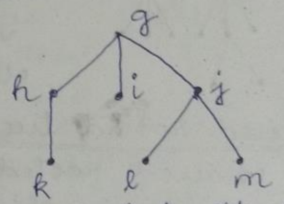
OMR Sheet No.__________ Registration No. 3

**Note:** i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

consisting of a and its descendants and all edges incident to these descendants.




Rooted tree T




Subtree rooted at g

**Eg** In the rooted tree T, find the parent of c, the children of g, the siblings of h, all ancestors of e, descendants of b, all internal vertices, and all leaves. What is the subtree rooted at g?

**Sol** Parent of c is b

Children of g are h, i and j

Siblings of h are i and j.

Ancestors of e are c, b, and a.

Descendants of b are c, d and e.

All internal vertices are a, b, c, g, h, and j

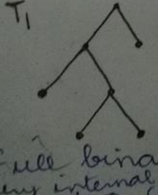
Leaves are f, d, e, i, k, l, m

The subtree rooted at g

**Def:** A rooted tree is called an m-ary tree if every internal vertex has no more than m children. [An m-ary tree with m = 2 is called a binary tree]. The tree is called a full m-ary tree if every internal vertex has exactly m children

**Eg** Are the rooted trees full m-ary trees for some positive integer m?

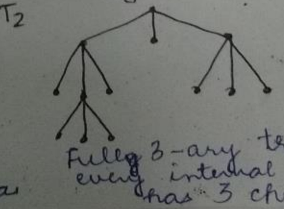
$T_1$



Full binary

Every internal vertex has two children

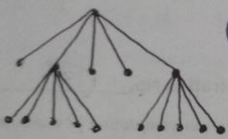
$T_2$



Full 3-ary trees

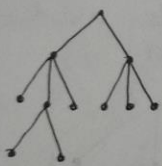
every internal vertex has 3 children.

$T_3$



(4) Every internal vertex has children so it is full 5-ary tree.

$T_4$



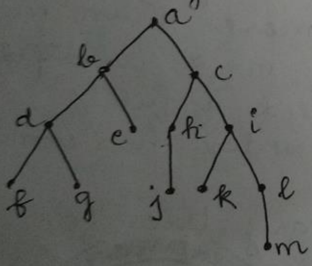
Not a full m-ary tree for any m because some of its internal vertices have two children and others have three children.

<u>Ordered rooted trees</u>

An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered. Here the children of each internal vertex are shown in order from left to right.

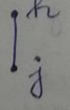
In an ordered binary tree, if an internal vertex has two children, the first child is called the <u>left child</u> and the second child is called the <u>right child</u>. The tree rooted at the left child of a vertex is called <u>left subtree</u> of this vertex, and the tree rooted at the right child of a vertex is called <u>right subtree</u> of the vertex.

<u>Eg</u> What are the left and right children of d in the binary tree? What are the left and right subtrees of c?

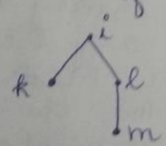



Left child of d is f and right child of d is g.

Leftsubtree of c




Right subtree of c



MR Sheet No.__________

Registration No. 5

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

## Properties of Trees

**Theorem:** A tree with $n$ vertices has $n-1$ edges

* If $G$ is connected and $G$ has $n-1$ edges, then $G$ has no simple circuits, so that $G$ is a tree.
* If $G$ has no simple circuits and $G$ has $n-1$ edges, then $G$ is connected and so is a tree
* If $G$ is connected and has no simple circuits then $G$ has $n-1$ edges and is a tree.

**Theorem:** A full $m$-ary tree with $i$ internal vertices contains $n = mi+1$ vertices.

**Theorem:** A full $m$-ary tree with

i) $n$ vertices has $i = \frac{n-1}{m}$ internal vertices and $l = \frac{(m-1)n+1}{m}$ leaves

ii) $i$ internal vertices has $n = mi+1$ vertices and $l = (m-1)i+1$ leaves

iii) $l$ leaves has $n = \frac{ml-1}{m-1}$ vertices and $i = \frac{l-1}{m-1}$ internal vertices.

eg. suppose that someone starts a chain letter. Each person who receives the letter is asked to send it on to 4 other people. some people do this, but others donot send any letters. How many people have seen the letter, including the first person, if no one receives more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out? How many people sent out the letter?

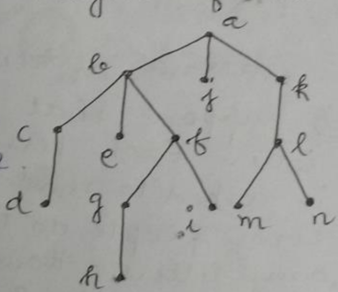
The chain letter can be represented usi 4-ary tree. The internal vertices corresp to people who sent out the letter, and the leaves correspond to people who didnot send it out. Because 100 people didnot send out the letter, $l = 100$. Hence

$$n = \frac{ml-1}{m-1} = \frac{4\times 100-1}{4-1} = \frac{399}{3} = 133.$$

There are total of 133 persons. 100 of them didnot send out. So, no. of internal vertices $= 133 - 100 = 33$ so 33 people sent out the letter.

* The level of a vertex $v$ in a rooted tree is the length of the unique path from the root to this vertex. The level of the root is defined to be zero. The height of a rooted tree is the length of the longest path from the root to any vertex.

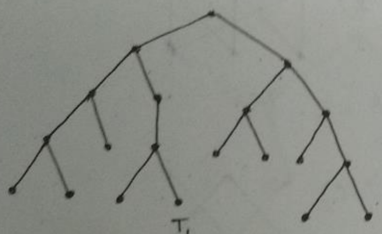
eg Find the level of each vertex in the rooted tree. What is the height of this tree?




Root a is at level 0.
Vertices b, j, k are at level 1
Vertices c, e, f, l are at level 2.
Vertices d, g, i, m, n are at level 3.
Vertex h is at level 4.
Because the largest level of any vertex is 4, this tree has height 4.

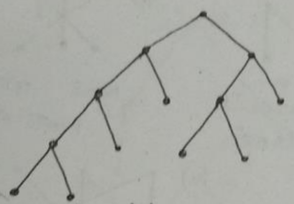
* A rooted m-ary tree of height $h$ is <u>balanced</u> if all leaves are at levels $h$ or $h-1$.

eg Which of the rooted trees are balanced?



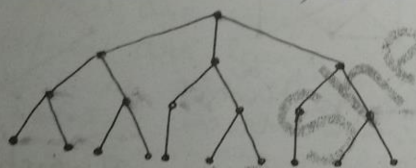
$T_1$

all leaves are at levels 3 or 4.
So, it is balanced.



$T_2$

all leaves are at levels 2, 3 and 4.
so, it is not balanced.



$T_3$

all leaves are at level 3 so $T_3$ is balanced.

* Theorem: There are at most $m^h$ leaves in an m-ary tree of height $h$.

* Corollary: If an m-ary tree of height $h$ has $l$ leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$
($\lceil x \rceil$ is the smallest integer greater than or equal to $x$)

* A complete m-ary tree is a full m-ary tree in which every leaf is at the same level.

* The eccentricity of a vertex in an unrooted tree is the length of the longest simple path beginning at this vertex. A vertex is called a center if no vertex in the tree has smaller eccentricity than this vertex.

* Chromatic number for a tree is two.

1) Which of these graphs are trees? (8)

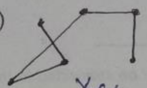
a)

Yes

b)

Not connected
Not tree

c)



Yes

d)

There is a simple circuit
Not a tree

e)

Not connected
Not a tree

f)

Yes

2) a)

Yes

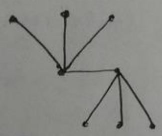
b)

Yes

c)

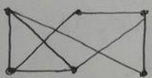
Not connected
Not a tree

d)



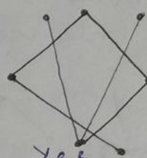
Tree.

e)



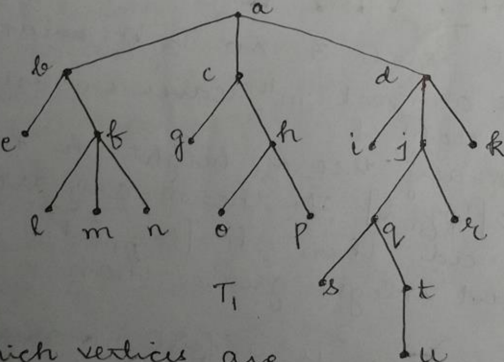
There is a simple circuit
∴ Not a tree

f)



Yes.

3)




a) Which vertex is the root?
a

b) Which vertices are internal?
a, b, c, d, f, h, j, q, t

c) Which vertices are leaves?
e, g, i, k, l, m, n, o, p, r, s, u.

d) Which vertices are children of j?
q and r

e) Which vertex is the parent of h? c

f) Which vertices are siblings of o? p

g) Which vertices are ancestors of m? f, b, a

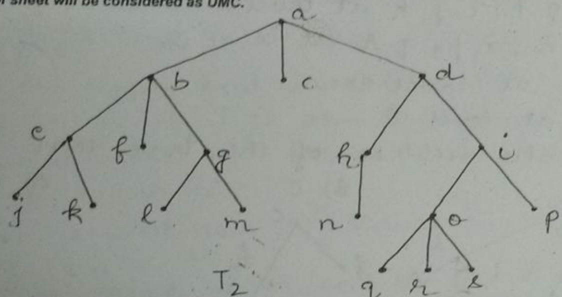
h) Which vertices are descendants of b? e, f, l, m, n

5) Is $T_1$ a full m-ary tree for some pos. integer m?
No, some vertices has two children, some have three ∴ so not a full m-ary tree.

OMR Sheet No.__________ Registration No. 9

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

4)




$T_2$

a) Which vertex is the root? a

b) Which vertices are internal? a, b, d, e, g, h, i, o

c) Which vertices are leaves? c, f, j, k, l, m, n, p, q, r, s

d) Which vertices are children of j? No child

e) Which vertex is the parent of h? d

f) Which vertices are siblings of o? p

g) Which vertices are ancestors of m? g, b, a

h) Which vertices are descendants of b? f, g, l, m

6) Is $T_2$ a full m-ary tree for some positive integer m?

No, every internal vertex does not have same no. of children.

8) What is the level of each vertex?

Vertex a at level 0

Vertices b, c, d at level 1

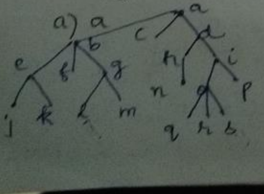
e, f, g, h, i at level 2

j, k, l, m, n, o, p at level 3

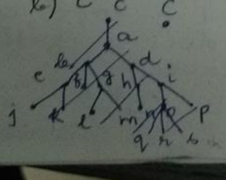
q, r, s at level 4

10) Draw the subtree of $T_2$ that is rooted at a

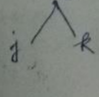
a) a



b) c c c



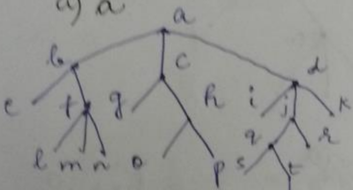
c) e



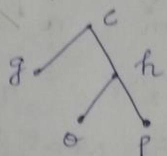
7) What is the level of each vertex of
Vertex a at level 0
Vertices b, c, d at level 1.
e, f, g, h, i, j, k at level 2.
l, m, n, o, p, q, r at level 3.
s, t at level 4.
u at level 5.

9) Draw the subtree of the tree that is rooted at

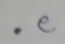
a) a




b) c




c) e




16) Which complete bipartite graphs $K_{m,n}$ where m and n are positive integers, are trees.

If m ~~or~~ and n $\geq 2$ then it forms a circuit of length 4 ∴ Not a tree.

If $m = 1$ or $n = 1$ then it forms a tree

$K_{1,n}$ or $K_{m,1}$ is a tree.

17) How many edges does ~~h~~ a tree with 10,000 vertices have?
no. of edges $= n - 1 = 10000 - 1 = 9999$ edges.

18) How many vertices does a full 5-ary tree with 100 internal vertices have?
$m = 5$, $i = 100$, $n = mi + 1$
$= 5 \times 100 + 1 = 501$ vertices

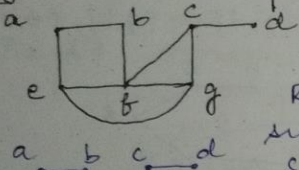
19) How many edges does a full binary tree with 1000 internal vertices have?
$m = \cancel{to}\ 2$, $i = 1000$, $n = mi + 1$
$= 2001$
edges $= n - 1 = 2000$ edges.

20) How many leaves does a full 3-ary tree with 100 vertices have?
$m = 3$, $n = 100$, $l = \frac{(m-1)n+1}{m} = \frac{2 \times 100 + 1}{3} = \frac{201}{3}$
$= 67$ leaves

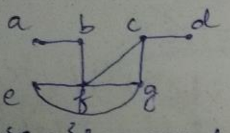
Spanning trees

Let G be a simple graph. A spanning tree of G that is a tree containing every vertese of G.

eg Find a spanning tree of simple graph

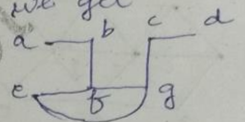



The graph is connected but it contains circuits. Remove {a, c}, we remove one simple circuit

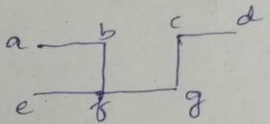



{a, c} removed

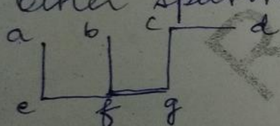
still it containe circuits. Now remove {c, f} we get a connected simple graph but there is a circuit e, f, g, e

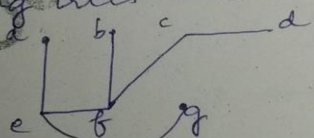


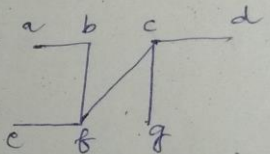

Now remove {e, g} we get a connected graph with no simple circuits ie tree. Hence it is a spanning tree of graph.




Other spanning trees are



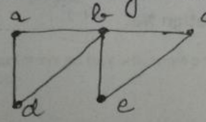

Theorem: A simple graph is connected iff it has a spanning tree.

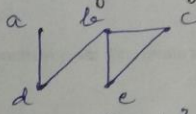
1) How many edges must be removed from a connected graph with n vertices and m edges to produce a spanning tree?

A tree with n vertices have $n-1$ edges so edges to be removed are $m-(n-1) = m-n+1$

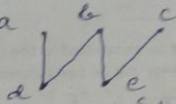
Find a spanning tree for the graph shown by removing edges in simple ci

2)





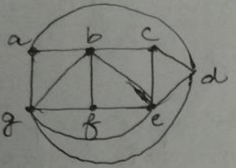
Remove{a, b}

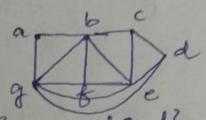


Remove {b, c}

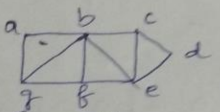
(13)

3)

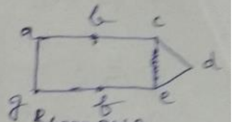




Remove {a, d}

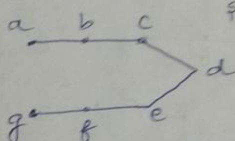


Remove {g, e}, {g, d}.

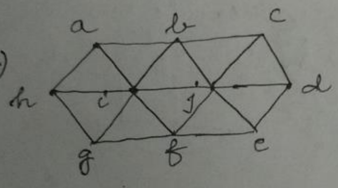


Remove {g, b}, {b, f}, {b, e}, {c, e

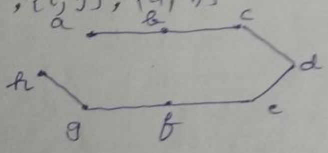
Remove {a, g}



4)



Remove {a, i}, {b, i}, {b, j}, {c,
{d, j}, {e, j} {f, j}, {f, i}, {g, i},
{h, i}, {i, j}, {a, h}



Sheet No.__________

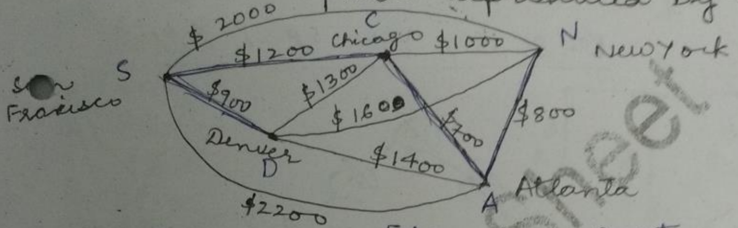
Registration No. (14)

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

Minimum spanning tree

A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Use Prim's Algorithm to design a min cost communications network connecting all the computers represented by the graph




| choice | Edge | cost |
|---|---|---|
| 1 | {Chicago, Atlanta} | $ 700 |
| 2. | {Atlanta, New York} | $ 800 |
| 3. | {Chicago, San Francisco} | $1200 |
| 4. | {San Francisco, Denver} | $900 |
| | | $3600 |

AC = 700 ✓
A-N = 800 ✓
A-D = 1400 X
A-S = 2200 X
C-N = 1000 X
C-D = 1300 X
C-S = 1200 ✓

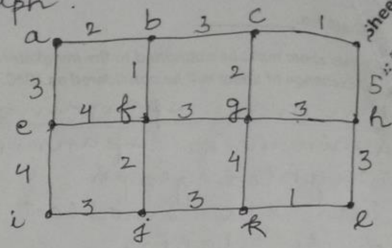
AN = 800
N-C = 1000 X
N-S = 2000 X
N-D = 1600 X

CS = 1200
S-N = 2000 X
S-D = 900 ✓
S-A = 2200 X

SD = 900

so min cost is $3600.

Q. Use Prim's algorithm to find a (15) mini[mum] spanning tree in the graph.




$\boxed{bf = 1}$ ✓

b — a = 2 ✓
  — c = 3 ✓

f — e = 4 ✗
  — j = 2 ✓
  — g = 3 ✗ cycle

$\boxed{ba = 2}$

a — e = 3 ✓

$\boxed{fj = 2}$

j — i = 3 ✓
  — k = 3 ✓

$\boxed{bc = 3}$

c — d = 1 ✓
  — g = 2 ✓

$\boxed{cd = 1}$

d — h = 5 ✗

$\boxed{cg = 2}$

g — f = 3 ✗ cycle
  — h = 3 ✓
  — k = 4 ✗

$\boxed{ji = 3}$

i — e = 4 ✗

$\boxed{jk = 3}$

k — g = 4 ✗
  — l = 1 ✓

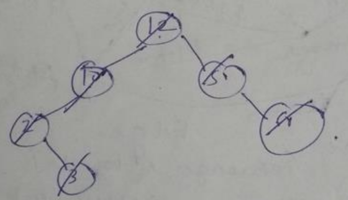
$\boxed{kl = 1}$

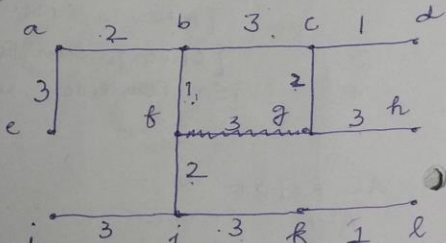
l — h = 3 ✗ cycle

$\boxed{gh = 3}$

h — d = 5 ✗
  — l = 3 ✗ cycle

10, 12, 13, 14, 5p, 6p






$\boxed{ae = 3}$

e — f = 4 ✗
  — i = 4 ✗

Min length = 24.

Sheet No.__________

Registration No. (16)

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

Eg Use Kruskal's algorithm to find a minimum spanning tree in the weighted graph

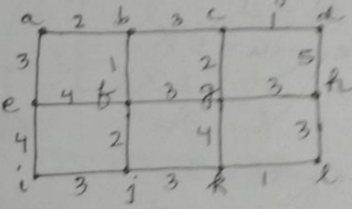
1 — bf ✓, cd ✓, kl ✓

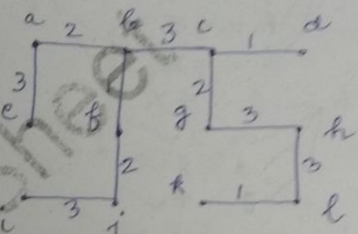
2 — ab ✓, cg ✓, fj ✓

3 — bc ✓, ae ✓, fg ✗, gh ✓, hl ✓, ij ✓, jk ✗

4 — cf ✗, ei ✗, gk ✗

5 — dh ✗



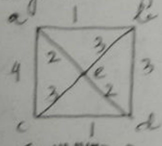

Minimum spanning tree

min ~~length~~ weight — 24



Exercises

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph

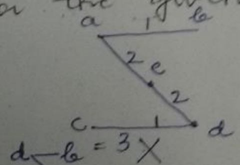



ab = 1

a — e = 2 ✓, c = 4 ✗

b — e = 3 ✗, d = 3 ✗

e — b = 3 ✗, c = 3 ✗, d = 2 ✓

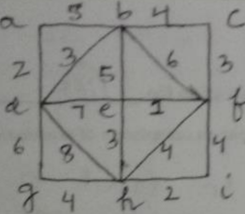



d — b = 3 ✗, c = 1 ✓

c — e = 3 ✗, a = 4 ✗

Minimum weight = 6

3)




ef = 1

e — b = 5 ✗
  — d = 7 ✗
  — h = 3 ✓

f — c = 3 ✓
  — b = 6 ✗
  — h = 4 ✗
  — i = 4 ✗

h — f = 4 ✗
  — i = 2 ✓
  — d = 8 ✗
  — g = 4 ✓

i — f = 4 ✗

c — b = 4 ✓

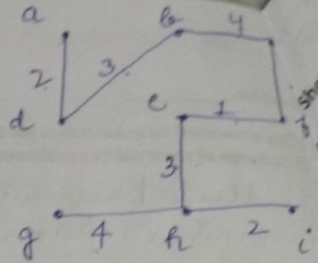
g — d = 6 ✗

b — e = 5 ✗
  — f = 6 ✗
  — d = 3 ✓
  — a = 5 ✗

d — a = 2 ✓
  — e = 7 ✗
  — h = 8 ✗
  — g = 6 ✗

a — b = 5 ✗

17




Minimum spanning tree

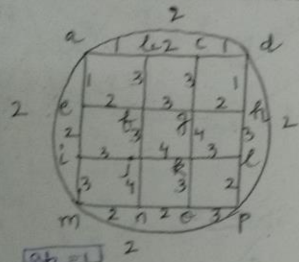
Min weight - 22

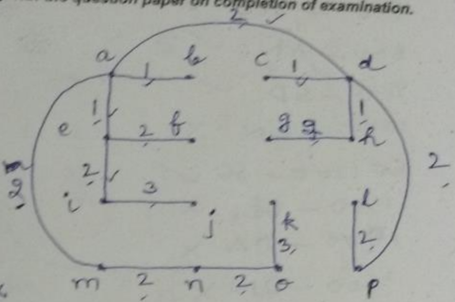
Sheet No.__________

Registration No. 18

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

4)





Min spanning tree

ab = 1

a ← d = 2 ✓
e = 1 ✓
m = 2 ✓

e ← f = 2 ✓
i = 2 ✓

i ← j = 3 ✓
m = 3 ✗

d ← c = 1 ✓
h = 1 ✓
p = 2 ✓

c ← b = 2 ✗
g = 3 ✗

~~h = g = 2~~

~~p ← l = 2~~
~~o = 3~~
~~m = 2~~

h ← g = 2 ✓
l = 3 ✗

m ← i = 3 ✗
n = 2 ✓
p = 2 ✗

p ← l = 2 ✓
o = 3 ✗
m = 2 ✗

f ← b = 3 ✗
g = 3 ✗
j = 3 ✗

b ← c = 2 ✗
f = 3 ✗

g ← c = 3 ✗
f = 3 ✗
k = 4 ✗
j = 4 ✗
o = 2 ✓

o ← k = 3 ✓
p = 3 ✗

l ← h = 3 ✗
k = 3 ✗

j ← f = 3 ✗
k = 4 ✗
n = 4 ✗

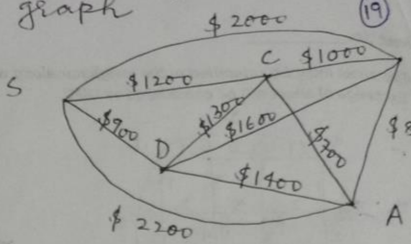
k ← l = 3 ✗
j = 4 ✗
g = 4 ✗

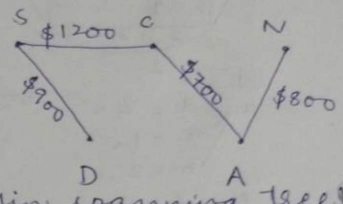
Min weight = 28

5) Use Kruskal's algorithm to design the comm network described in graph

~~700~~ — AC ✓
800 — AN ✓
900 — SD ✓
1000 — CN ×
~~SC~~ 1200 — SC ✓
1300 — DC ×
1400 — DA ×
1600 — DN ×
2000 — SN ×
2200 — SA ×

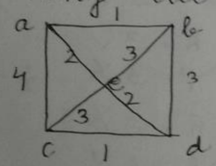
Min weight = \$3600

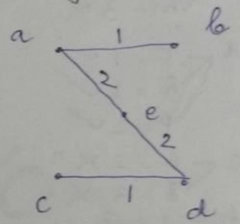



Min spanning tree

6) Use Kruskal's algorithm to find a min spanning tree for weighted graph



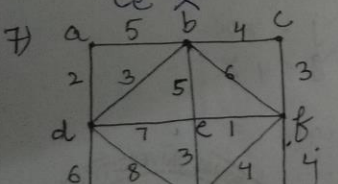



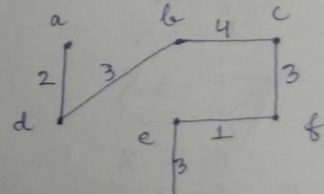

1 — ab ✓, cd ✓
2 — ae ✓, ed ✓
3 — eb ×, bd ×, ce ×
4 — ac ×

Min spanning tree

Min weight = 6.

7)



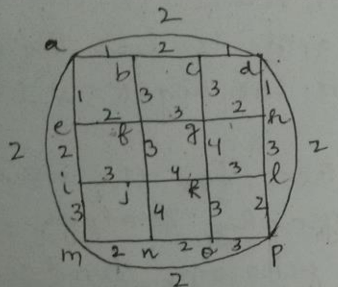

1 — ef ✓
2 — ad ✓, hi ✓
3 — bd, cf, eh
4 — bc ✓, hf ×, fi ×, gh ✓
5 — ab ×, be ×
6 — bf ×, dg ×
7 — de ×
8 — dh ×

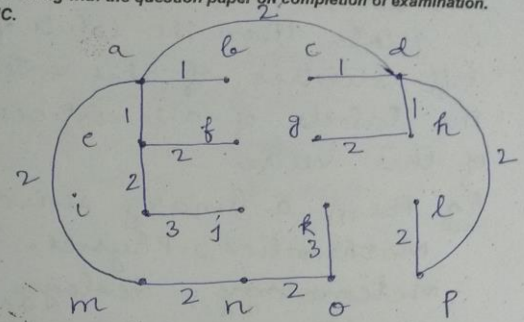
(19)

Sheet No.__________ Registration No. (20)

Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.

8)




Minimum spanning tree

Minimum weight = 28

1 — ab ✓
cd ✓
ae ✓
dh ✓

2 — ad ✓, am ✓, mn ✓
bc X, df, no ✓
ef ✓, mp X, pl ✓
gh ✓, ei ✓

3 — bf X, cg X, fj X
fg X, ij ✓, kl X
hl X, im X, ko ✓
op X

4 — kj X, kg X
jn X

# Binary Search Trees

(21)

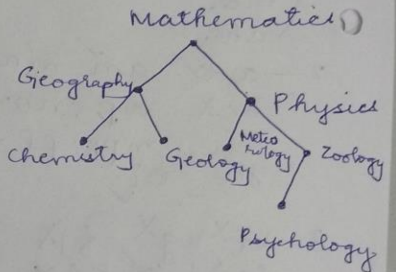
It is a binary tree in which the vertices a labeled with items so that a label of a verte is greater than the labels of all vertices in the left subtree of this vertex and is less than the labels of all vertices in the right subtre of this vertex

eg Form a binary search tree for the words Mathematics, Physics, Geography, Zoology, Meteorology, Geology, Psychology and Chemistry (using alphabetical order)

Sol Let the key be Mathematics

M < P

Mathematics < Physics




G < M

Geography < Mathematics

Now P < Z

Physics < Zoology

M < P

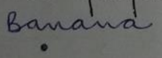
Meteorology < Physics

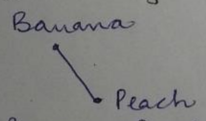
Geography < Geology

Psychology < Zoology

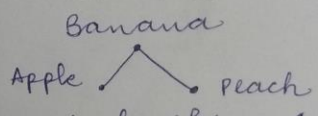
Chemistry < Geography

1) Build a binary search tree for the words banana, peach, apple, pear, coconut, mango and papaya using alphabetical order.

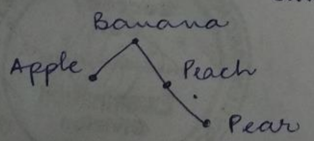



Banana < Peach

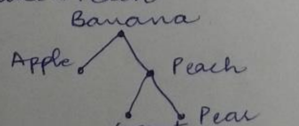



Apple < Banana



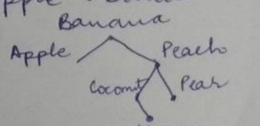

Banana < Pear

Peach < Pear




Banana < Coconut

Cocunut < Peach




Banana < Mango

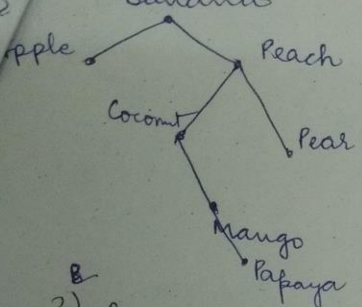
Mango < Pap

Coconut < Mango

Sheet No.__________

Registration No. 22

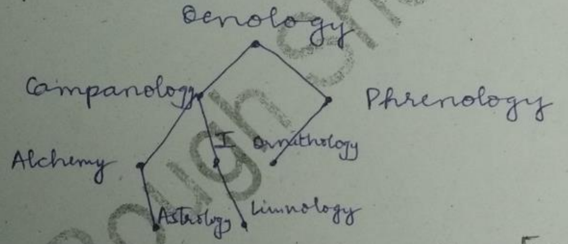
Note: i) This sheet must be submitted to the invigilator along with the question paper on completion of examination.
ii) Exchange of sheet will be considered as UMC.



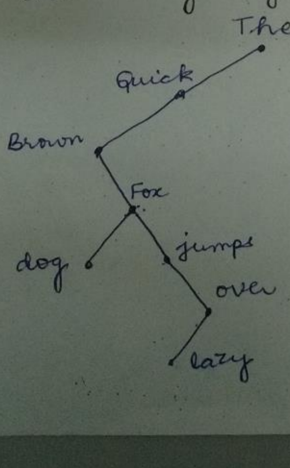

Banana < Papaya
Papaya < Peach
Coconut < Papaya
Mango < Papaya

8
2) Build a binary search tree for the words Oenology, phrenology, campanology, ornithology, Ichthyology, limnology, alchemy, astrology using alphabetical order

Sol




5) Using alphabetical order, construct a binary search tree for the words in the sentence "The quick brown fox jumps over the lazy dog."



3, 12, 8, 15, 7, 9, 1, 2